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ソボレフ空間による関数の近似について (再生核の理論の応用)

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ソボレフ空間による関数の近似について

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Abstract. Let $H_K(E)$ be a reproducing kernel Hilbert space comprising complex-valued functions $\{f\}$ on E and L_j ($j = 1, 2, \dots$) be a bounded linear operator on $H_K(E)$ into a Hilbert space H_j . Then, for $d_j \in H_j$ we shall consider the simultaneous operator equations $L_j f = d_j$ ($j = 1, 2, \dots$) with the best approximation problem, for given $d_j \in H_j$

$$\inf_{f \in H_K(E)} \sum_j \|L_j f - d_j\|_{H_j}^2.$$

Furthermore we shall give a general idea and method for approximations of L_2 functions by Sobolev Hilbert spaces by using the Tikhonov regularization. We shall illustrate examples by figures for approximations of L_2 functions by the first and second order Sobolev Hilbert spaces.

Keywords: Reproducing kernel, operator equations, bounded linear operator, Tikhonov regularization, Sobolev space, best approximation, Green's function, simultaneous linear partial differential equation, generalized inverse.

1 Introduction and Background Theorems

We shall formulate our background theorem which has many concrete applications based on [2-6].

Let H_K be a Hilbert space comprising complex-valued functions $\{f\}$ on a set E admitting a reproducing kernel $K(x, y)$ and let L be a bounded linear

operator on H_K into a Hilbert space H . We introduce the inner product in the space H_K , for any fixed $\lambda > 0$

$$\lambda(f_1, f_2)_{H_K} + (Lf_1, Lf_2)_H. \quad (1)$$

Then, it forms a Hilbert space and this Hilbert space $H_K(L; \lambda)$ admits a reproducing kernel $K_L(x, y; \lambda)$ on E . Then, we have the relation of $K(x, y)$ and $K_L(x, y; \lambda)$

$$K_L(x, y; \lambda) + \frac{1}{\lambda}(LK_L(\cdot, y; \lambda), LK(\cdot, x))_H = \frac{1}{\lambda}K(x, y). \quad (2)$$

Theorem 1 *The best approximation f_{λ, g, f_0}^* in the sense, for any $f_0 \in H_K$ and for any $g \in H$*

$$\begin{aligned} & \inf_{f \in H_K} \left\{ \lambda \|f - f_0\|_{H_K}^2 + \|Lf - g\|_H^2 \right\} \\ &= \lambda \|f_{\lambda, g, f_0}^* - f_0\|_{H_K}^2 + \|Lf_{\lambda, g, f_0}^* - g\|_H^2 \end{aligned} \quad (3)$$

exists uniquely and it is represented by

$$f_{\lambda, g, f_0}^*(x) = \lambda(f_0(\cdot), K_L(\cdot, x; \lambda))_{H_K} + (g(\cdot), LK_L(\cdot, x; \lambda))_H. \quad (4)$$

As simple and typical reproducing kernel Hilbert spaces, we shall consider the Sobolev Hilbert spaces H_{K_1} and H_{K_2} admitting the reproducing kernels

$$K_1(x, y) = \frac{1}{2}e^{-|x-y|} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\xi(x-y)}}{\xi^2 + 1} d\xi \quad (5)$$

and

$$K_2(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\xi(x-y)}}{\xi^4 + \xi^2 + 1} d\xi. \quad (6)$$

The norms in H_{K_1} and H_{K_2} are given by

$$\|f\|_{H_{K_1}}^2 = \int_{-\infty}^{\infty} (|f'(x)|^2 + |f(x)|^2) dx$$

and

$$\|f\|_{H_{K_2}}^2 = \int_{-\infty}^{\infty} (|f''(x)|^2 + |f'(x)|^2 + |f(x)|^2) dx,$$

respectively. We shall examine the best approximation in (3) for some typical Hilbert spaces H and bounded linear operators L . In general, we are interested in the behaviours of the best approximation functions for λ tending to zero from the viewpoint of the Tikhonov regularization. So, we wish to illustrate the behaviours of the best approximations for λ tending to zero.

2 Typical Examples

See [3] for many concrete reproducing kernel forms for which Theorem 1 is applied. We can see a general example and a general approach for simultaneous linear partial differential equations in N. Aronszajn [1] who discussed deeply Green's functions in connection with reproducing kernels. We shall give typical examples.

2.1 Let

$$G(x, y) = \frac{1}{2}e^{-|x-y|}. \quad (7)$$

Then $G(x, y)$ is the reproducing kernel for the Hilbert Sobolev space H_G comprising all absolutely continuous functions $f(x)$ on \mathbf{R} with finite norms

$$\left\{ \int_{-\infty}^{\infty} (|f'(x)|^2 + |f(x)|^2) dx \right\}^{\frac{1}{2}} < \infty. \quad (8)$$

Hence, we can examine the best approximation problem as follows:

For any given $F_1, F_2 \in L_2(\mathbf{R})$,

$$\inf_{f \in H_G} \int_{-\infty}^{\infty} (|F_1(x) - f'(x)|^2 + |F_2(x) - f(x)|^2) dx. \quad (9)$$

2.2 For the first order Sobolev Hilbert space H_{K_1} we shall consider the two bounded linear operators $L_1 : H_{K_1} \rightarrow L_2(\mathbf{R})$ and $L_2 : H_{K_1} \rightarrow L_2(\mathbf{R})$. Then, the associated reproducing kernels $K_{1,1}(x, y; \lambda)$ and $K_{1,2}(x, y; \lambda)$ for the RKHSs with the norms

$$\lambda \|f\|_{H_{K_1}}^2 + \|f\|_{L_2(\mathbf{R})}^2$$

and

$$\lambda \|f\|_{H_{K_1}}^2 + \|f'\|_{L_2(\mathbf{R})}^2$$

are given by

$$\begin{aligned} K_{1,1}(x, y; \lambda) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\xi(x-y)}}{\lambda\xi^2 + (\lambda+1)} d\xi \\ &= \frac{1}{2\sqrt{\lambda(\lambda+1)}} \exp \left\{ -\sqrt{\frac{\lambda+1}{\lambda}} |x-y| \right\} \end{aligned} \quad (10)$$

and

$$K_{1,2}(x, y; \lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\xi(x-y)}}{(\lambda+1)\xi^2 + \lambda} d\xi$$

$$= \frac{1}{2\sqrt{\lambda(\lambda+1)}} \exp \left\{ -\sqrt{\frac{\lambda}{\lambda+1}} |x-y| \right\}, \quad (11)$$

respectively. Hence, the best approximate functions $f_{1,1}^*(x; \lambda, g)$ and $f_{1,2}^*(x; \lambda, g)$ in the senses, for any $g \in L_2(\mathbf{R})$

$$\begin{aligned} & \inf_{f \in H_{K_1}} \left\{ \lambda \|f\|_{H_{K_1}}^2 + \|f - g\|_{L_2(\mathbf{R})}^2 \right\} \\ &= \lambda \|f_{1,1}^*(\cdot; \lambda, g)\|_{H_{K_1}}^2 + \|f_{1,1}^*(\cdot; \lambda, g) - g\|_{L_2(\mathbf{R})}^2 \end{aligned} \quad (12)$$

and

$$\begin{aligned} & \inf_{f \in H_{K_1}} \left\{ \lambda \|f\|_{H_{K_1}}^2 + \|f' - g\|_{L_2(\mathbf{R})}^2 \right\} \\ &= \lambda \|f_{1,2}^*(\cdot; \lambda, g)\|_{H_{K_1}}^2 + \|f_{1,2}^{*'}(\cdot; \lambda, g) - g\|_{L_2(\mathbf{R})}^2 \end{aligned} \quad (13)$$

are given by

$$f_{1,1}^*(x; \lambda, g) = \int_{-\infty}^{\infty} g(\xi) \frac{1}{2\sqrt{\lambda(\lambda+1)}} \exp \left\{ -\sqrt{\frac{\lambda+1}{\lambda}} |\xi - x| \right\} d\xi \quad (14)$$

and

$$f_{1,2}^*(x; \lambda, g) = \int_{-\infty}^{\infty} g(\xi) \frac{1}{2\sqrt{\lambda(\lambda+1)}} \frac{\partial}{\partial \xi} \exp \left\{ -\sqrt{\frac{\lambda}{\lambda+1}} |\xi - x| \right\} d\xi, \quad (15)$$

respectively. Note that $f_{1,2}^*(x; \lambda, g)$ can be considered as an approximate and generalized solution of the differential equation

$$y' = g(x) \quad \text{on } \mathbf{R} \quad (16)$$

in the first order Sobolev Hilbert space H_{K_1} . See Figure 3.

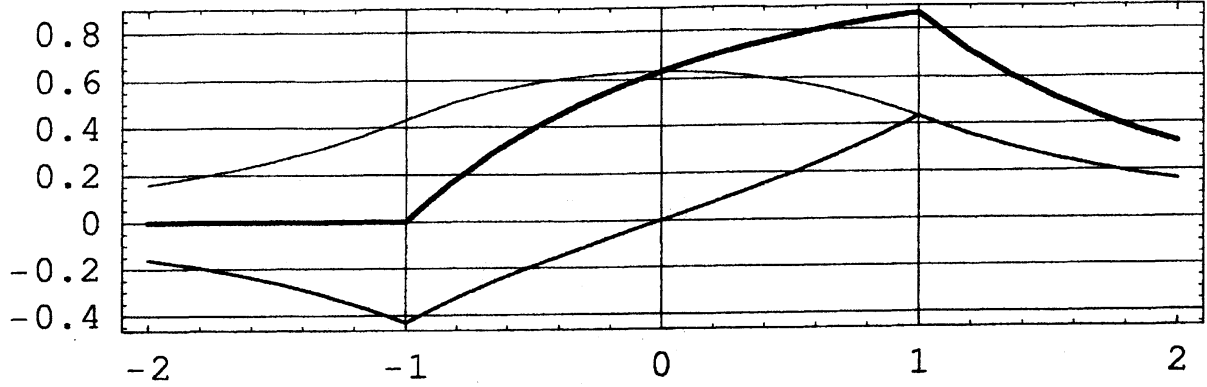


Figure 1: Examples of approximated functions in (9). (a) $F_1(x) = 0$ and $F_2(x) = \chi_{[-1,1]}$ (top thin curve). (b) $F_1(x) = F_2(x) = \chi_{[-1,1]}$ (middle bold curve). (c) $F_1(x) = \chi_{[-1,1]}$ and $F_2(x) = 0$ (bottom curve).

2.3 For the second order Sobolev Hilbert space H_{K_2} we shall consider the three bounded linear operators into $L_2(\mathbf{R})$ defined by

$$L_1 : f \longrightarrow f$$

$$L_2 : f \longrightarrow f'$$

and

$$L_3 : f \longrightarrow f''.$$

Then, the reproducing kernels $K_{2,1}(x, y; \lambda)$, $K_{2,2}(x, y; \lambda)$ and $K_{2,3}(x, y; \lambda)$ for the Hilbert spaces with the norms

$$\lambda \|f\|_{H_{K_2}}^2 + \|f\|_{L_2(\mathbf{R})}^2,$$

$$\lambda \|f\|_{H_{K_2}}^2 + \|f'\|_{L_2(\mathbf{R})}^2,$$

and

$$\lambda \|f\|_{H_{K_2}}^2 + \|f''\|_{L_2(\mathbf{R})}^2,$$

are given by

$$K_{2,1}(x, y; \lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\xi(x-y)}}{\lambda \xi^4 + \lambda \xi^2 + (\lambda + 1)} d\xi, \quad (17)$$

$$K_{2,2}(x, y; \lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\xi(x-y)}}{\lambda \xi^4 + (\lambda + 1)\xi^2 + \lambda} d\xi, \quad (18)$$

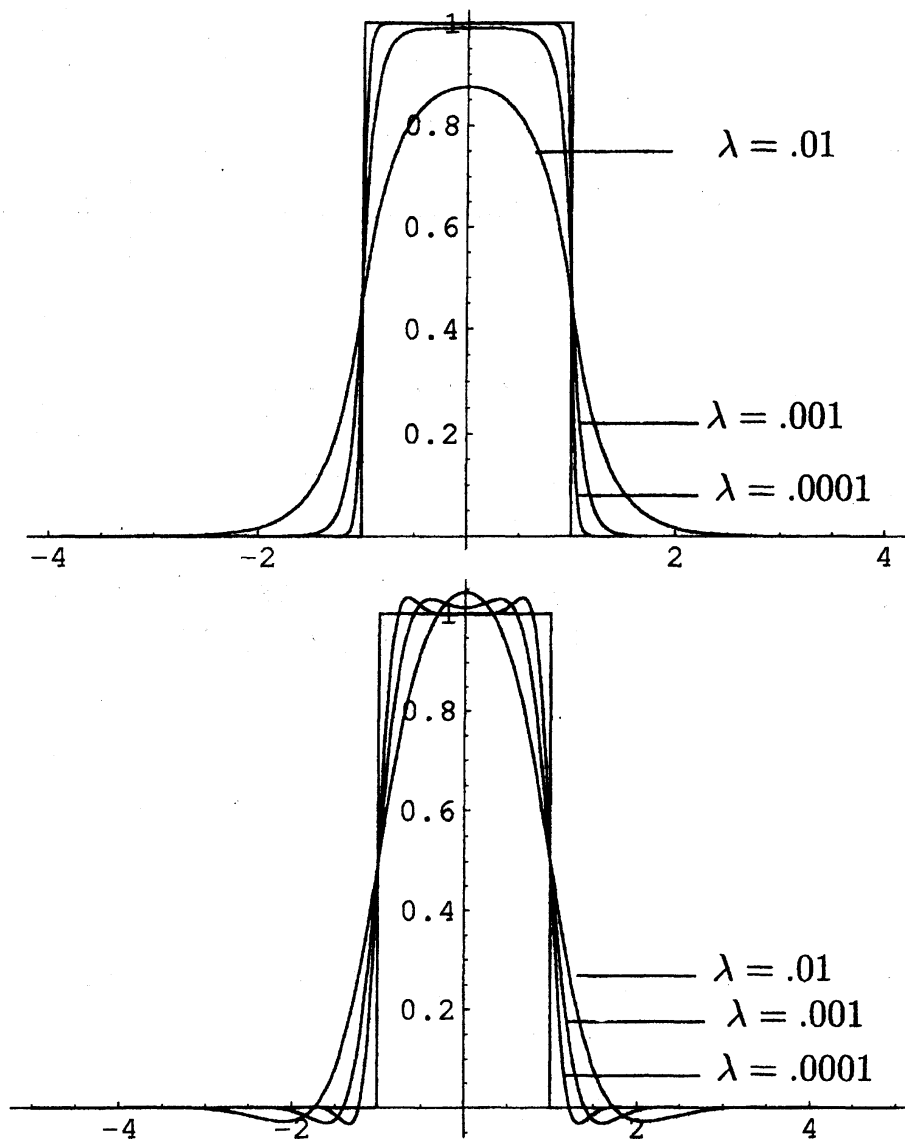


Figure 2: Graphs of $f_{1,1}^*(x; \lambda, g)$ in (14)(top) and $f_{2,1}^*(x; \lambda, g)$ in (20) (bottom) for $g(x) = \chi_{[-1,1]}$.

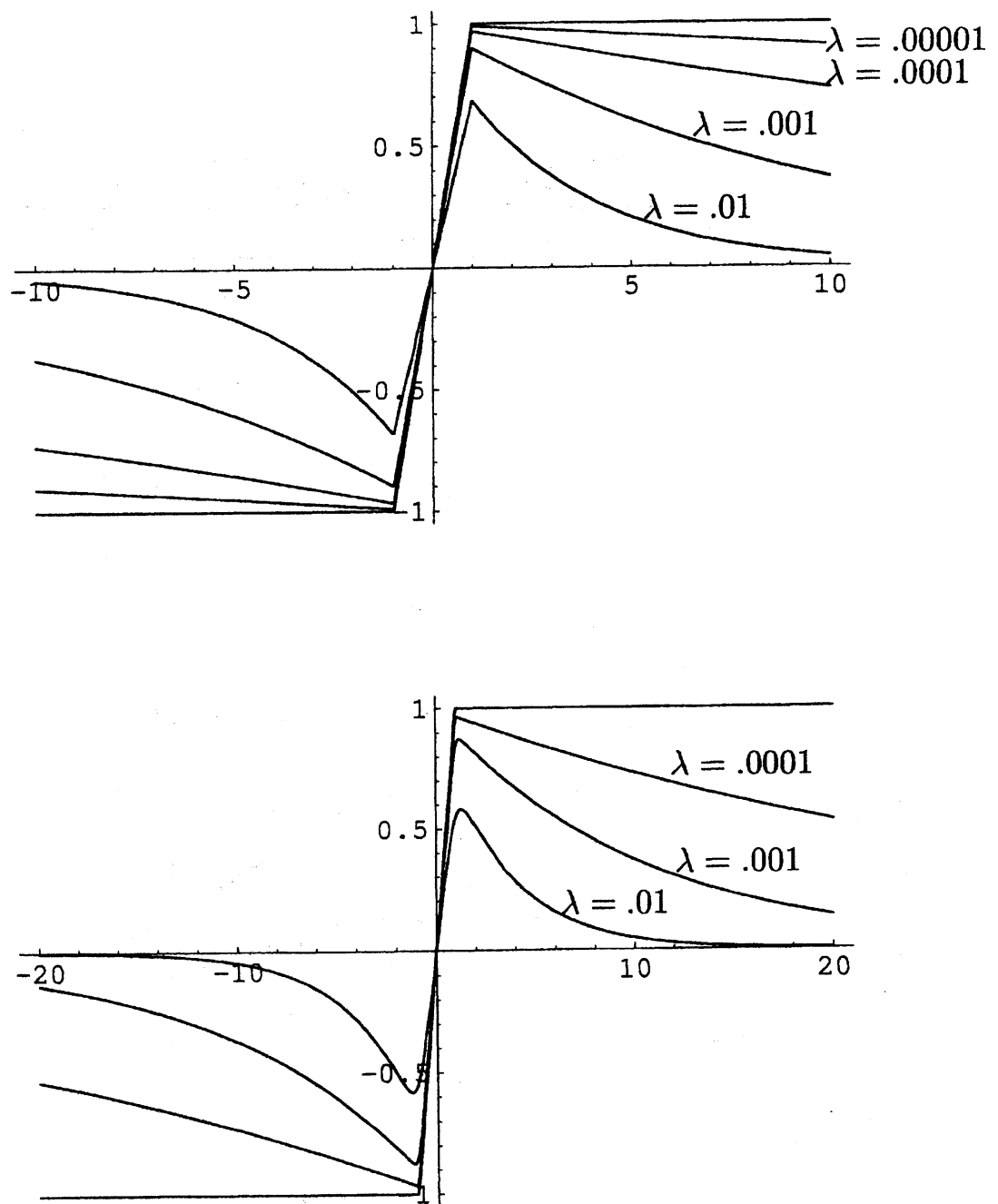


Figure 3: Graphs of $f_{1,2}^*(x; \lambda, g)$ in (15) (top) and $f_{2,2}^*(x; \lambda, g)$ in (21) (bottom) for $g(x) = \chi_{[-1,1]}$.

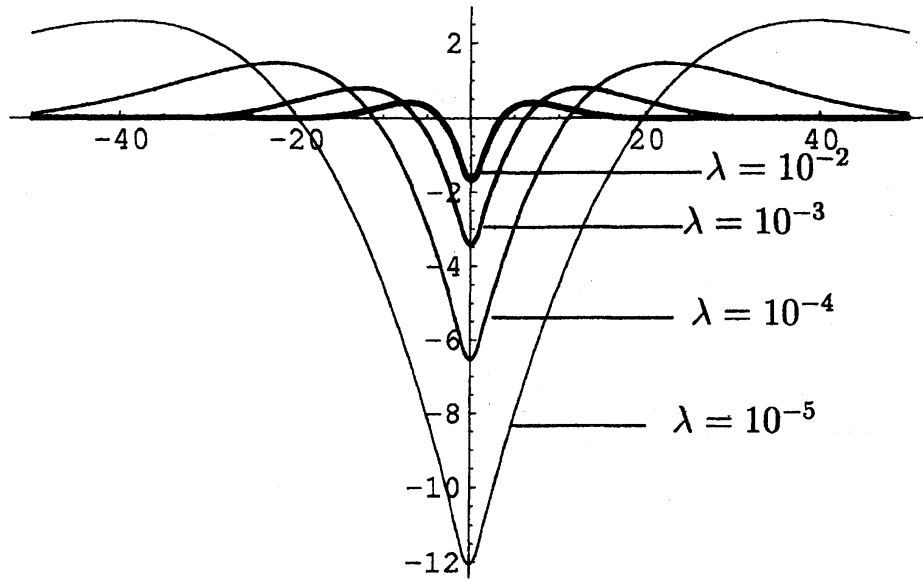


Figure 4: Graphs of $f_{2,3}^*(x; \lambda, g)$ in (22) for $g(x) = \chi_{[-1,1]}$.

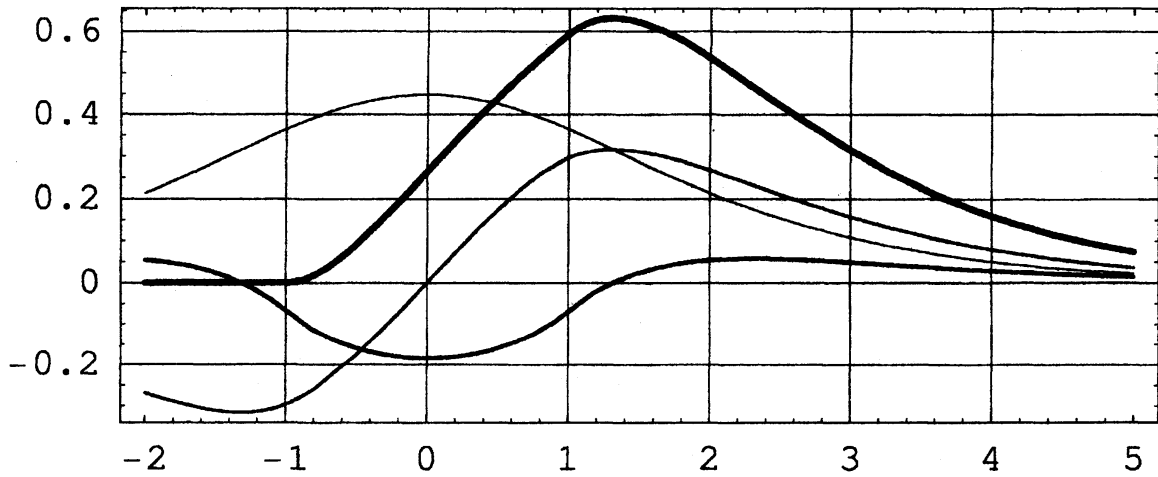


Figure 5: Examples of approximated functions in (25). (a) $F_1(x) = F_2(x) = F_3(x) = \chi_{[-1,1]}$ (top bold curve). (b) $F_1(x) = \chi_{[-1,1]}$ and $F_2(x) = F_3(x) = 0$ (the second bold curve) (c) $F_1(x) = F_2(x) = 0$ and $F_3(x) = \chi_{[-1,1]}$ (the thin curve). (d) $F_1(x) = F_3(x) = 0$ and $F_2(x) = \chi_{[-1,1]}$ (the rest curve).

and

$$K_{2,3}(x, y; \lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\xi(x-y)}}{(\lambda+1)\xi^4 + \lambda\xi^2 + \lambda} d\xi. \quad (19)$$

Then, the corresponding best approximate functions $f_{2,1}^*(x; \lambda, g)$, $f_{2,2}^*(x; \lambda, g)$, and $f_{2,3}^*(x; \lambda, g)$ are given by, for any $g \in L_2(\mathbf{R})$

$$f_{2,1}^*(x; \lambda, g) = \int_{-\infty}^{\infty} g(\xi) d\xi \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\eta(\xi-x)}}{\lambda\eta^4 + \lambda\eta^2 + (\lambda+1)} d\eta, \quad (20)$$

$$f_{2,2}^*(x; \lambda, g) = \int_{-\infty}^{\infty} g(\xi) d\xi \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-i\eta \cdot e^{-i\eta(\xi-x)}}{\lambda\eta^4 + (\lambda+1)\eta^2 + \lambda} d\eta, \quad (21)$$

and

$$f_{2,3}^*(x; \lambda, g) = \int_{-\infty}^{\infty} g(\xi) d\xi \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-\eta^2 \cdot e^{-i\eta(\xi-x)}}{(\lambda+1)\eta^4 + \lambda\eta^2 + \lambda} d\eta, \quad (22)$$

respectively. We shall give another type applications of Theorem 1. Note that

$$K(x, y) = \frac{1}{4} e^{-|x-y|} \{1 + |x-y|\} \quad (23)$$

is the reproducing kernel of the Sobolev space H_K with finite norms

$$\left\{ \int_{-\infty}^{\infty} (|f''(x)|^2 + 2|f'(x)|^2 + |f(x)|^2) dx \right\}^{\frac{1}{2}} < \infty. \quad (24)$$

Therefore, we can examine the approximate problem as follows:

$$\inf_{f \in H_K} \int_{-\infty}^{\infty} (|F_1(x) - f''(x)|^2 + 2|F_2(x) - f'(x)|^2 + |F_3(x) - f(x)|^2) dx. \quad (25)$$

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